A DETERMINANT FORMULA FOR THE NUMBER OF WAYS OF COLORING A MAP.

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Suppose that a finite set of two-dimensional regions making up a simply or multiply connected closed surface are given, so that these form a *map* M. Each of these regions may be taken to be limited by closed curves, formed by a finite number of continuous *boundary lines* which the region has in common with other regions. The ends of these lines, at which three or more regions meet, are called *vertices* of the map. A *coloring* of the map consists in attributing to each region a color different from that of any region having in common with it a boundary line, but not necessarily different from that of a region meeting it at a vertex.

The following fact will first be proved: *The number of ways of coloring the given map M in λ colors* ($\lambda = 1, 2, \cdots$) *is given by a polynomial $P(\lambda)$ of degree n, where n is the number of regions of the map M.* In fact let $m_i (i = 1, 2, \cdots, n)$ be the number of ways of coloring the map by using exactly i colors *when mere permutations of the colors are disregarded.* With this definition it is clear that

$$m_i \cdot \lambda \cdot (\lambda - 1) \cdots (\lambda - i + 1)$$

represents the number of ways of coloring the given map in exactly i of the λ colors *counting two colorings as distinct when they are obtained by a permutation one from the other*; for, of the i colors used, the first may be chosen in λ ways, the second in $\lambda - 1$ ways, and so on. If λ is less than i the above term reduces to zero.

But the total number of ways of coloring the given map in λ colors is the sum of the number of ways of coloring it with $1, 2, \cdots, n$ of these colors, since no more than n colors can be used. Accordingly the total number of ways is represented by

$$P(\lambda) = m_1\lambda + m_2\lambda(\lambda - 1) + \cdots + m_n\lambda(\lambda - 1) \cdots (\lambda - n + 1)$$

for all values of λ. It is clear that in general $m_1 = 0$ inasmuch as for $n > 1$ no map can be colored in a single color, and that $m_n = 1$ since there is only one way of coloring M in n colors if permutation of the colors be disregarded.

In order to proceed to the effective determination of $P(\lambda)$ we consider the total number $\mu - 1$ of ways of forming from the map $M^{(1)} = M$ *submaps* $M^{(2)}, \cdots, M^{(k)}$ of $n-1$ regions, $M^{(k+1)}, \cdots, M^{(l)}$ of $n-2$ regions, and so on to $M^{(\mu)}$ of one region, by successive coalescence of regions adjacent along a boundary line. Such a coalescence may be indicated by the removal of all the common boundary lines of the two regions which coalesce. The maps $M^{(2)}, \cdots, M^{(k)}$ are obtained by one such step, the maps $M^{(k+1)}, \cdots, M^{(l)}$ by two such steps, and so on.

At this point we introduce the symbol (i, k) to denote the number of ways of breaking down the map M in n regions to a submap of i regions by k simple or multiple coalescences, i. e., by picking out maps $M, M^{(a_1)}, \cdots, M^{(a_k)}$, each but the first being a submap of the preceding one, and the last one having i regions. It is apparent that we have $(i, k) = 0$ for $k > n - i$, and that $(i, n-i)$ represents the number of ways of making $n - i$ successive simple coalescences. By definition we take $(n, 0) = 1$ and $(i, 0) = 0$ for $i < n$.

Let now one of the λ colors be placed at random on each of the regions of any map $M^{(i)}$ of the μ maps above defined. Each one of these arrangements will color *one and one only* of the maps $M^{(i)}$ and its submaps $M^{(i_1)}$, $M^{(i_2)}, \cdots$, namely that one obtained by a coalescence of all adjacent regions which receive the same colors. In consequence if we let $\sigma_1, \sigma_2, \cdots, \sigma_\mu$ denote the number of ways of coloring $M^{(1)}, M^{(2)}, \cdots, M^{(\mu)}$ respectively in λ colors, we will have

$$\lambda^{n_i} = \sigma_i + \sigma_{i_1} + \cdots \qquad (i = 1, 2, \cdots, \mu),$$

in which the symbol n_i denotes the number of regions in $M^{(i)}$, and λ^{n_i} is then the total number of ways of giving one of the λ colors to each region; on the right appear the numbers $\sigma_i, \sigma_{i_1}, \sigma_{i_2}, \cdots$ corresponding to $M^{(i)}$ and its submaps $M^{(i_1)}, M^{(i_2)}, \cdots$. Let ϵ_{ij} for $i \neq j$ be 1 or 0 according as $M^{(i)}$ does or does not contain $M^{(j)}$ as a submap, and let ϵ_{ii} be 1; we may write the above equations in the form

$$\lambda^{n_i} = \sum_{j=1}^{\mu} \epsilon_{ij}\sigma_j \qquad (i = 1, 2, \cdots, \mu).$$

This set of μ equations is linear in the μ quantities $\sigma_1, \cdots, \sigma_\mu$. If we observe that because of the arrangement of the submaps of M according to a decreasing number of regions we have $i_1 > i, i_2 > i, \cdots$ in the above equations, it becomes clear that $\epsilon_{ij} = 0$ for $i > j$. Thus the determinant of this system of equations is 1 and therefore solving for $\sigma_1 = P(\lambda)$ we obtain

$$P(\lambda) = \begin{vmatrix} \lambda^{n_1}, & \epsilon_{12}, & \cdots, & \epsilon_{1n} \\ \lambda^{n_2}, & \epsilon_{22}, & \cdots, & \epsilon_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda^{n_\mu}, & \epsilon_{\mu 2}, & \cdots, & \epsilon_{\mu n} \end{vmatrix}$$

a determinant formula for the number of ways of coloring the map in λ colors.

This determinant is of the order μ, approximately of the magnitude n !. Furthermore it might be proved that the quantities ϵ_{ij} determine the constitution of the map, so that if the determinant were written in full the structure of the complete map might be deduced from it. These two facts show the complicated character of the determinant.

The evaluation of this determinant may be carried out in terms of the symbols (i, k) previously introduced. To this end let us consider a typical term

$$\pm \lambda^{n_j} \epsilon_{\alpha 2} \epsilon_{\beta 3} \cdots \epsilon_{\kappa\mu}$$

where the $-$ or $+$ sign is taken according as $j, \alpha, \beta, \cdots, \kappa$ gives an odd or even permutation of $1, 2, \cdots \mu$, i. e., according as the substitution

$$\begin{pmatrix} 1 & 2 & \cdots & \mu \\ j & \alpha & \cdots & \kappa \end{pmatrix}$$

is the product of an odd or an even number of transpositions.

Any such term either reduces to zero or to $\pm \lambda^{n_j}$. We shall consider how terms not zero may arise.

If the above substitution is not the identical substitution it may always be decomposed into a product of cyclic substitutions $(\rho_1, \rho_2, \cdots, \rho_k)$ composed of k elements and changing ρ_1 to ρ_2, ρ_2 to ρ_3, $\cdots$, ρ_k to ρ_1. Such a cyclic substitution must contain the element 1; else there arises in the term a product of factors

$$\epsilon_{ab}, \ \epsilon_{bc}, \ \cdots, \ \epsilon_{la}$$

which is zero necessarily since we cannot have simultaneously $a < b$, $b < c, \cdots, l < a$. *It follows that the substitution degenerates into a single cyclic substitution at most, containing the element* 1.

The corresponding term is thus of the form

$$\pm \lambda^{n_j} \epsilon_{ja} \epsilon_{ab} \cdots \epsilon_{l1} \epsilon_{mm} \epsilon_{pp} \cdots,$$

where the $+$ or $-$ sign is to be taken according as the cyclic substitution $(1, j, a, \cdots, l)$ contains an odd or an even number of elements. Conversely to every product of this sort which is not zero we have a single term not zero of the determinant.

Suppose now that we attempt to obtain the sum of all the terms of this kind for a given $n_j = i$ and a given number of elements $k + 1$ of this cyclic substitution. If none of the factors ϵ_{ja}, ϵ_{ab}, $\cdots$ are to be zero the map $M^{(j)}$ contains $M^{(a)}$ as a submap, the map $M^{(a)}$ contains $M^{(b)}$ as a submap, and so on. Thus we obtain a sequence of $k+1$ maps M, $M^{(j)}$, $M^{(a)}, \cdots, M^{(l)}$, each after the first a submap of the preceding one. Consequently there is one and only one such term corresponding to each way of breaking down M in k steps to some submap of i regions, and each such term has the same sign as $(-1)^k$.

The stated terms therefore are $(-1)^k(i, k)\lambda^i$ in value and the final formula for the number of ways of coloring the given map in λ colors is

$$P(\lambda) = \sum_{i=1}^{n} \lambda^i \sum_{k=0}^{n-i} (-1)^k (i, k).$$

The term in λ^n, corresponding to the identical substitution, has the proper coefficient unity according to our previous convention by which the symbol $(n, 0)$ has the value 1.

As a first example of the formula we take the very simple case of a map of three regions which are adjacent each to each. We will have

$$(2, 1) = 3, \quad (1, 1) = 1, \quad (1, 2) = 3,$$

and

$$P(\lambda) = (3, 0)\lambda^3 - (2, 1)\lambda^2 + [-(1, 1) + (1, 2)]\lambda = \lambda(\lambda - 1)(\lambda - 2).$$

The validity of this formula may be verified at once by noticing that we can color any one of these three regions in λ colors, a second region in the $\lambda - 1$ remaining colors, and the third region in the $\lambda - 2$ colors left after the first two regions are colored.

As a second example we take the case of a map in five regions formed by a ring of three regions bounding an interior and exterior region. In this case the symbols (i, k) which enter have the values

$$\begin{array}{llll} (4, 1) = 9, & (3, 1) = 22, & (3, 2) = 51; & \\ (2, 1) = 14, & (2, 2) = 125, & (2, 3) = 150; & \\ (1, 1) = 1, & (1, 2) = 45, & (1, 3) = 176, & (1, 4) = 150 \end{array}$$

so that

$$P(\lambda) = \lambda^5 - 9\lambda^4 + 29\lambda^3 - 39\lambda^2 + 18\lambda = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^2$$

In this case also the validity of the formula may be at once verified, for the three regions of the ring must be in three distinct colors, while the interior and exterior regions may be in any fourth color different from these three colors.

Even in this second case the value of the symbols (i, k) is not immediately obtained; and if we have a somewhat more complicated map, for example the map formed by twelve five-sided regions on the sphere, a considerable computation would be necessary to determine $P(\lambda)$ directly from the formula, or from the map itself.

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