

New open questions related to old conjectures by Tor Helleseth

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Introduction

In the two/three last years, progress in the direction of two conjectures by Helleseeth (1976) regarding the cross-correlation of maximal sequences have been obtained. The goal of this talk is to present several new interesting open questions over finite fields related to these conjectures.

- slides location :

<http://langevin.univ-tln.fr/recherche/drafts/openpb.pdf>

Fourier coefficient

Let L be a finite field of characteristic p and order q . The *Fourier coefficient* of a polynomial $f \in L[X]$ at a point $a \in L$ is

$$\widehat{f}(a) = \sum_{x \in K} \mu(f(x) - ax)$$

and more generally, for $b \in K$:

$$\widehat{f}_b(a) = \sum_{x \in K} \mu(bf(x) - ax)$$

where μ is the canonical additive character of K .

Remark

The minus sign that appears in the definition of the Fourier coefficient is not usual but there are several good reasons to adopt it.

Convolution

Let F, G two complex mappings over L .

$$G * F(t) = \sum_{y+x=t} G(x)F(y)$$

Denoting $F: x \mapsto \mu(f(x))$, and $\mu_a: x \mapsto \mu(ax)$:

$$\mu_a * F = \widehat{f}(a) \mu_a$$

- μ_a is an eigenvector
- $\widehat{f}(a)$ eigenvalue.

Spectrum

For a permutation f of L

$$\text{spec}(f) = \{\widehat{f}(a) \mid a \in L^\times\}.$$

Definition

The permutation f is said to be r -valued, where $r = \#\text{spec}(f)$,

$$D(f) = \prod_{a \in L^\times} \widehat{f}(a)$$

Power permutation

We are interested by the **power permutations**:

$$x \mapsto f(x) = x^s, \quad (s, q-1) = 1.$$

[equivalence]

$$s' \sim s \iff \exists j, s' = sp^j \pmod{q-1}$$

Remark

Like for any permutation π , the phase Fourier coefficient of any power permutation is null,

$$\widehat{\pi}(0) = \sum_{x \in L} \mu(\pi(x)) = \sum_{x \in L} \mu(x) = 0.$$

Invariance of the Fourier distribution

Note that for a power permutation

$$\forall b \in K^\times, \quad \text{spec}(f_b) = \text{spec}(f)$$

because

$$\widehat{f}_b(a) = \sum_x \mu(bx^s - ax) = \widehat{f}(ab^{-1/s})$$

Problem (invariance by translation)

What are the maps f such that

$$\forall b \in K^\times, \quad \text{spec}(f_b) = \text{spec}(f)?$$

Basic arithmetic facts

Let $\zeta_p = \exp(2i\pi/p)$, $\wp = (1 - \zeta_p)$ the prime ideal above p in $\mathbb{Z}[\zeta_p]$, σ_t the Galois automorphism of $\mathbb{Q}[\zeta_p]$ defined by $\sigma_t(\zeta_p) = \zeta_p^t$.

For a power permutation f :

$$\widehat{f}(a) \equiv \widehat{f}(0) \equiv 0 \pmod{\wp}, \quad \sigma_t(\widehat{f}(a)) = \widehat{f}_t(at)$$

Lemma (action of Galois)

If s is invertible then $\text{spec}(x^s)$ is invariant by the Galois group of $\mathbb{Q}(\zeta_p)$.

Lemma (integrality)

All the Fourier coefficients of x^s are integral iff $s \equiv 1 \pmod{p-1}$.

- Hellesth, 1976.

Valuation of the Fourier coefficients

For an exponent s , we define

$$V_K(s) = \min_{a \in K} \text{val}_p(\widehat{f}(a))$$

This parameter is connected to Stickelberger congruences on Gauss sum,

$$\widehat{f}(a) = \frac{q}{q-1} + \frac{1}{q-1} \sum_{\chi \neq 1} \tau_K(\chi) \tau_K(\bar{\chi}^s) \chi^s(a)$$

thus

$$V_K(s) = \min_{1 \neq \chi \in K^\times} \text{val}_p(\tau_K(\chi) \tau_K(\bar{\chi}^s))$$

using **Hasse-Davenport**, given an extension L/K :

$$V_L(s) \leq V_K(s) \times [L : K]$$

Helleseth vanishing conjecture

A permutation f of L is *singular*

$$\exists a \in L^\times, \widehat{f}(a) = 0, \quad \text{i.e. } D(f) = 0$$

Conjecture (Helleseth conjecture I)

All the power permutations $x \mapsto x^s$ with $s \equiv 1 \pmod{p-1}$ are singular.

Hard ?! May be false ?!

Very difficult to progress on this question.

- A numerical evidence checked for $[L : \mathbb{F}_2] \leq 25$ [PL, 2007].
- If $p = 2$ then 3 divides $D(x^s)$ [Yves Aubry, PL, 2013]
- If $[L : \mathbb{F}_2] = \ell^r$ then $\ell | D(x^s)$
- True for a 3-valued exponent [Daniel Katz, 2012].

Boolean challenges

Problem

find a direct proof for $s = -1$, i.e. Kloostermann sum.

Problem

find an 5-adic analogue of AL result.

Problem

find new general divisibility results.

Problem

find an analogue of Katz result , for 4-valued exponents !

3-valued power permutations

Theorem (Daniel Katz, 2012)

A 3-valued power permutation x^s is *singular* and

$$s = 1 \pmod{p-1}, \quad \text{spec}(x^s) = \{0, A, B\} \subset \mathbb{Z}.$$

Moreover, the number of solutions of

$$x + y = 1, \quad x^s + y^s = 1$$

is equal to

$$V = A + B - \frac{AB}{q}$$

Folklore Calderbank, Blokuis.

Coefficient of the Minimal polynomial

More generally, the product

$$P(f) = \prod_{0 \neq A \in \text{spec}(f)} A$$

the rational number $P(f)/q$ appears naturally by Fourier analysis.

Problem (valuation of coefficients)

It is true that q divides the $P(f)$?

Helleseth 3-valued conjecture

Conjecture (Helleseth conjecture II)

If $[L : \mathbb{F}_p]$ is a power of two then the spectrum of an invertible exponent is not three valued.

- Tao Feng (2012) : $p = 2$ assuming annulation of the spectrum.
- Proved for $p \leq 3$, Daniel Katz (2014).

key point

Tao Feng uses the following proposition to obtain Helleseth conjecture II in even characteristic assuming the singularity of a 3-valued exponents.

Proposition (Calderbank, McGuire, Poonen, Rubinstein, 1996)

Let $s \neq 1$ be an invertible exponent. If $[K : \mathbb{F}_2]$ is a power of two then

$$2 \times V_K(s) \leq [K : \mathbb{F}_2].$$

Remark

In fact the same holds for all p !!!

quadratic extension

Lemma (Yves Aubry, Daniel Katz, PL)

Let L/K be a quadratic extension. If x^s is constant over K^\times but not over L then

$$\exists a \in L, \quad \widehat{f}(a) = -|K|, \quad 2 \times V_L(s) = [L : \mathbb{F}_p]$$

Using Hasse-Davenport relation, we now see by **induction** that if

$$1 \not\equiv s \equiv 1 \pmod{p-1} \quad \text{and} \quad [L : \mathbb{F}_p] = 2^r$$

then

$$2 \times V_L(s) \leq [L : \mathbb{F}_p].$$

Notation

From now and on, s is a three valued invertible exponent : it takes three values 0, A , and B over a finite field L of order $q = p^m$, p prime. Note that s is congruent to 1 modulo $(p - 1)$.

$$A = p^a \alpha, \quad B = p^b \beta, \quad A - B = p^c \gamma$$

with α , β and γ coprime with p .

differential multiplicity

Let us denote by $N(u, v)$ the number of solutions of the system

$$\begin{cases} x + y = u \\ f(x) + f(y) = v \end{cases}$$

By Fourier analysis

$$N(u, v) = \frac{1}{q^2} \sum_{a,b} \widehat{f}_b(a)^2 \mu(au - bv)$$

differential exponent

Definition

Let s be an exponent. We say that s is a Δ -differential exponent over K if the number of solutions of

$$\begin{cases} x + y = 1, \\ x^s + y^s = v, \end{cases}$$

is equal to 0 or Δ for all $v \neq 1$.

3-valued exponent

Following the argumentation of Tao Feng

Theorem (Katz)

If s is three valued over L then $\alpha\beta\gamma$ divides the differential multiplicities $N(1, v)$ for all $v \neq 1$ and

$$|\alpha\beta\gamma| \leq -\frac{AB}{q}$$

leading to the alternative

- 1 $2V_L(s) > [L : \mathbb{F}_p]$ (*impossible when $[L : \mathbb{F}_p] = 2^r$*)
- 2 $2V_L(s) = [L : \mathbb{F}_p]$
 $\gamma = 1$ and s is $|\alpha\beta|$ -differential exponent.

Corollary

if $p = 2$ or $p = 3$ then Helleseth conjecture II is true.

Nice exponent

Definition

Let s be an exponent. We say that s is a *nice exponent* over K if the number of solutions of

$$\begin{cases} x + y = 1, \\ x^s + y^s = v, \end{cases}$$

takes at most 3 values.

Remark

A Δ -differential exponent is nice.

Numerical result

It is easy to find all the differential distribution of all exponents using Zech logarithm.

Let ω be a primitive root of K :

$$\text{Zech}(k) = l, \quad 1 + \omega^k = \omega^l$$

the logarithm of

$$x^s + (1 - x)^s = x^s \left(1 + \left(\frac{1-x}{x}\right)^s\right)$$

for $x = \omega^k$ is

$$k \times s + \text{Zech}[(\text{Zech}[n + k] - k) \times s], \quad n := \frac{q-1}{2}.$$

<http://langevin.univ-tln.fr/project/>

sample

```
[pl@microbe ~]$ cat ~/web-docs/project/expo/nice-11.txt
#field is GF(11,2)
#field is GF(11,3)
    3   : 3 : 3 : 664 [ 0]    1 [ 1] 665 [ 2]
   447  : 7 : 3 : 664 [ 0]    1 [ 1] 665 [ 2]
  1209  : 9 : 3 : 664 [ 0]    1 [ 1] 665 [ 2]

#field is GF(11,4)
   241  : 1 : 3 : 7379 [ 0] 7260 [ 2]    1 [121]
  4921  : 1 : 3 : 7379 [ 0] 7260 [ 2]    1 [121]

#field is GF(11,5)
    3   : 3 : 3 : 80524 [ 0]    1 [ 1] 80525 [ 2]
   53687 : 7 : 3 : 80524 [ 0]    1 [ 1] 80525 [ 2]
 146409 : 9 : 3 : 80524 [ 0]    1 [ 1] 80525 [ 2]
```

New conjectures ?

Conjecture (nice exponent)

Asuming odd characteristic. Let s be an exponent. If s is nice then 2 is a differential multiplicity.

Conjecture (optimist)

Asuming odd characteristic. Let s be an exponent. If s is invertible then 2 is a differential multiplicity.

optimist \implies nice \implies Helleseth 3-valued



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