Spectral Moment of Order Four and the Uniqueness of the CCZ class of Dublin APN Permutation

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Almost Perfect Nonlinear maps

Let $F : \mathbb{F}_2^m \to \mathbb{F}_2^m$, for $u, v \in \mathbb{F}_2^m$,

$$N(u,v) = \text{number of solutions of } \begin{cases} x + y = u, \\ F(x) + F(y) = v. \end{cases}$$

$$\delta(F) = \max\{N(u,v) : u \neq 0,\ v \in \mathbb{F}_2^m\} \geqslant 2.$$

APN

F is APN $\iff \delta(F) = 2$

cube

x^3 is APN and it is a permutation iff the dimension m is odd

remember good news from Ireland at F_{q^9} !

John Dillon Waterville, Kerry the week before Fq9. . . Kim Browning, Mike Macquistan, Adam Wolfe.

The Big APN Problem

- **2006** — *Xiang-Dong Hou* no APN permutation in dimension 4.
- **2009** — *Dillon and al.* Dublin APN-permutation in 6 variables.

open problem

Existence of APN permutation in **even** dimension greater than 6 ?

open question

Do we know all APN in dimension **6** ?

CCZ-equivalence

Graph of $F : \mathbb{F}_2^m \to \mathbb{F}_2^n$

$$\Gamma(F) = \{(x, F(x)) \mid x \in \mathbb{F}_2^m\} \subseteq \mathbb{F}_2^m \times \mathbb{F}_2^n$$

$$F \underset{\text{CCZ}}{\sim} G \quad \Leftrightarrow \quad \exists\, \mathfrak{T} \in \mathrm{AGL}(m+n), \quad \mathfrak{T}\big(\Gamma(F)\big) = \Gamma(G)$$

Remark

If F is a permutation then $F \underset{\text{CCZ}}{\sim} F^{-1}$

Compatibility

If F is APN and $F \underset{\text{CCZ}}{\sim} G$, then G is APN

EA-equivalence

Two (m, n)-vectorial functions F and G are EA-equivalent if

$$G = A \circ F \circ B + C$$

for some $A \in \mathrm{AGL}(n)$, $B \in \mathrm{AGL}(m)$, and $C \in \mathrm{Aff}(m, n)$.

EA is finer than CCZ

$$F \underset{\text{EA}}{\sim} G \Longrightarrow F \underset{\text{CCZ}}{\sim} G$$

$$\deg_{\min}(F) := \min_{F \underset{\text{CCZ}}{\sim} G} \deg(G) \quad \deg_{\max}(F) := \max_{F \underset{\text{CCZ}}{\sim} G} \deg(G)$$

Known APN classes in 6 variables

14 CCZ classes of APN splitting in **716 EA classes**

all but one have a quadratic representative, all have maximal degree 4.

14 CCZ class of known APN														
# **EA**	25	19	91	19	3	13	91	3	92	85	85	91	13	86

— *minimal degree 3*

— *class of Dublin permutation*

open question

Is there another CCZ class in 6 variables ?

What did one observed ?

- no APN of degree 6 is known
- no APN of degree 5 is known
- most of APN have degree 4
- all APN of degree 3 are known

$$\sharp\text{quartics} = 2^{90} \times \sharp\text{cubics}$$

open question

is there an APN CCZ class purely quartic i.e. minimal degree 4 ?

spectral moment of order 4

Let f be a Boolean function

$$f : \mathbb{F}_2^m \to \mathbb{F}_2$$

Walsh coefficient :

$$\forall a \in \mathbb{F}_2^m, \quad \hat{f}(a) = \sum_{x \in \mathbb{F}_2^m} (-1)^{f(x)+a\cdot x}.$$

normalized spectral moment of order 4 (kurtosis) :

$$\kappa(f) = \frac{1}{q^3} \sum_{a \in \mathbb{F}_2^m} \hat{f}(a)^4, \quad q = 2^m.$$

spectral caracterisation

F is APN iff

$$\sum_{0 \neq f \in \langle F \rangle} \kappa(f) = 2(q-1).$$

degree	2	3	4	5	6
♯class	1	3	62	335	686
cumul.	1	4	66	401	1087

number of class of 6-bit Boolean functions with $\kappa(f) \leq 2$

spectral level

Let F be an (m, n)-map.

$$F = (f_1, f_2, \ldots, f_n)$$

component spaces of F :

$$\langle F \rangle := \mathrm{span}(f_1, f_2, \ldots, f_n)$$

spectral set of F :

$$\mathfrak{K}(F) := \{\kappa(f) \mid 0 \neq f \in \langle F \rangle\}$$

The cardinality of $\mathfrak{K}(F)$ is the spectral level of F.

2-spectral level

If the dimension m is even then $\kappa(f) \neq 2$

The spectral level of a 6-bit APN map satisfies $\sharp\mathfrak{K}(F) \geq 2$

A vectorial APN-function F has *2-spectral levels* of type (α, β) when

$$\sharp\mathfrak{K}(F) = \{\alpha, \beta\}, \quad \alpha < \beta.$$

$$\alpha A + \beta B = 2(q-1), \quad A + B = q - 1;$$

$$A := \sharp\{f \in \langle F \rangle \mid \kappa(f) = \alpha\} \qquad B := \sharp\{f \in \langle F \rangle \mid \kappa(f) = \beta\}$$

most of quadratic APN have type (1,4).

One can prove that a 6-bit APN of type (1,4) is quadratic !

Analysis of Dublin permutation

0	54	48	13	15	18	53	35
25	63	45	52	3	20	41	33
59	36	2	34	10	8	57	37
60	19	42	14	50	26	58	24
39	27	21	17	16	29	1	62
47	40	51	56	7	43	44	38
31	11	4	28	61	46	5	49
9	6	23	32	30	12	55	22

Table – Permutation of Dublin has type $(1.75, 4)$

Observation

- 2 spectral levels, $\alpha = 1.75$ (56 comp) and $\beta = 4$ (7 comp),
- 7 components in a 3-dimensional subspace.

In fact the Boolean components splits in exactely two EA-class !

Numerical exploration

In dimension 6, backtracking approachs are sufficientely effective to construct APN-extensions of a given (6,4)-vectorial mapping.

1. 2-switching of known APN — done
2. extension of $(6, 3)$-bent — bent project
3. spectral moment of order 4 — ab-apn project

methods that generate large sets of $(6, 4)$-vectorial mappings that are candidates for APN-extension.

- invariant in (1)+(2)+(3)
- APN-extentibility test (2)+(3)

2-flat condition

Given a vectorial (m, n)-maps G, we define the set of 2-flats :

$$\mathfrak{F}_G := \left\{\text{2-flat } \{x, y, z, t\} \mid G(x) + G(y) + G(z) + G(t) = 0\right\}$$

$$x + y + z + t = 0$$

$$F \text{ is APN} \qquad \Longleftrightarrow \qquad \mathfrak{F}_F = \emptyset \ .$$

The extension $F := (G, g)$ of $G \in \mathrm{Vect}(m, m-2)$ by $g \in \mathrm{Vect}(m, 2)$ is APN iff :

$$\forall \{x, y, z, t\} \in \mathfrak{F}_G \qquad g(x) + g(y) + g(z) + g(t) \neq 0.$$

Looking g as a mapping into the field $\mathbb{F}_4$:

$$g(x) + g(y) + g(z) + g(t) \neq 0 \Leftrightarrow \big(g(x) + g(y) + g(z) + g(t)\big)^3 = 1$$

APN extension test

Looking g as a mapping into the field $\mathbb{F}_4$:

$$g(x)+g(y)+g(z)+g(t)\neq 0 \Leftrightarrow \big(g(x)+g(y)+g(z)+g(t)\big)^3=1$$

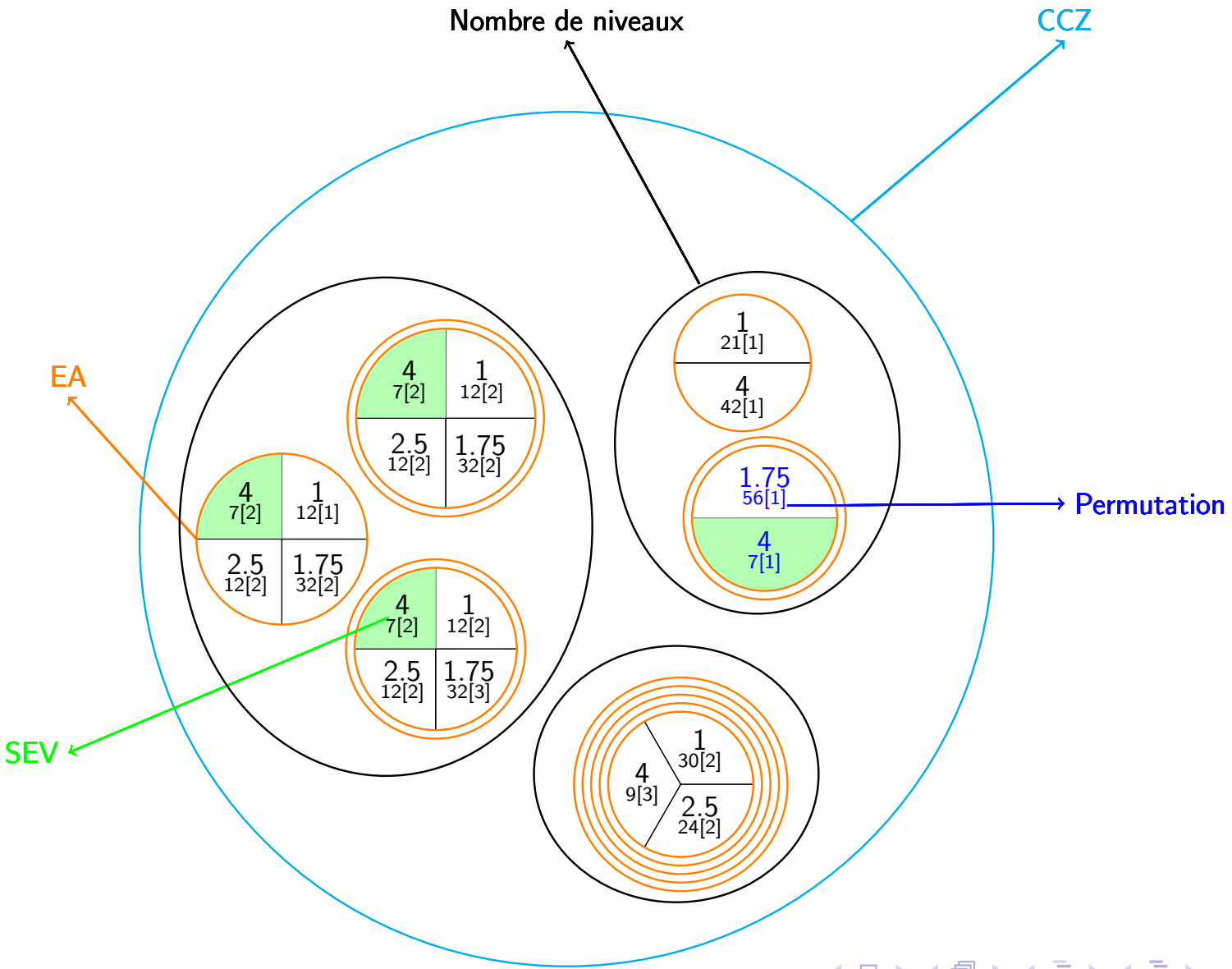
We indroduce in $q(q+1)/2 := q+q(q-1)/2$ *boolean variables* :

$$[x] := g(x)^3, \quad [x,y] := g(x)g(y)\big(g(x)+g(y)\big)$$

$$[x]+[y]+[z]+[t]+[x,y]+[x,z]+[x,t]+[y,z]+[y,t]+[z,t]=1.$$

no solution $\Longrightarrow$ not APN-extendable !

Dissection of CCZ-class of Dublin



challenge

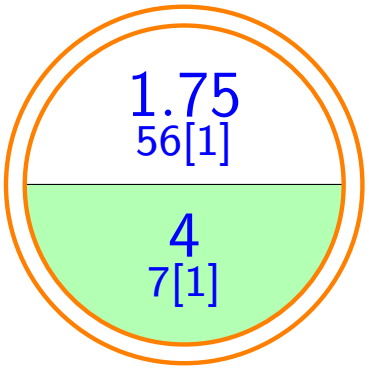
What are the CCZ-classes sharing the same shape that Dublin permutation ?

$$\mathfrak{K}(F) = \{1.75, 4\}$$

and $\langle F \rangle$ contains

- 3-dimensional space of cubics with $\kappa = 4.0$
- 56 quartics with $\kappa = 1.75$

Method of attack



- enumerate all $(6, 4)$-quartic satisfying (*)
- keep the maps that are APN-expandable
- find APN-expansion by backtraking
- extract 2-spectral level function
- compare with Dublin permutation

(*) 8 quartics $\kappa = 1.75$, 7 cubics $\kappa = 4.0$.

key points

The procedure depends EA-invariant algorithm !

- Only eight EA-classes of quartics with $\kappa = 1.75$
- A fast algorithm to check APN-expandibility
- 1 week of computation using 72 cores

At the end :

Uniqueness of the CCZ class of Dublin Permutation

other pairs involving spaces ?

α	A	degree	classe	β	B	degree	classe
1.0000	(56)	23...	4	10.0000	(7)	.3...	1
1.0000	(15)	23...	4	2.3125	(48)	.34..	214
1.0000	(7)	23...	4	2.1250	(56)	..4..	49
1.7500	(56)	..4..	8	4.0000	(7)	234..	86
1.9375	(56)	..4..	54	2.5000	(7)	.34..	216
1.9375	(62)	..4..	54	5.8750	(1)	..4..	19

ab-APN project

α	A	deg	♯	β	B	deg	♯
1.0	42	23.	4	4.0	21	234	86
1.0	60	23.	4	22.0	3	.3.	1
1.0	56	23.	4	10.0	7	.3.	1
1.0	49	23.	4	5.50	14	.34	29
1.0	21	23.	4	2.50	42	.34	216
1.0	57	23.	4	11.50	6	.34	5
1.0	35	23.	4	3.250	28	.34	191
1.0	15	23.	4	2.3125	48	.34	214
1.0	51	23.	4	6.250	12	..4	13
1.0	47	23.	4	4.9375	16	..4	37
1.0	39	23.	4	3.6250	24	..4	67
1.0	7	23.	4	2.1250	56	..4	49
1.0	55	23.	4	8.8750	8	..4	2

α	A	deg	♯	β	B	deg	♯
1.750	56	..4	8	4.0	7	234	86
1.750	42	..4	8	2.50	21	.34	216
1.750	60	..4	8	7.0	3	.34	3
1.750	51	..4	8	3.0625	12	.34	321
1.750	35	..4	8	2.3125	28	.34	214
1.750	59	..4	8	5.6875	4	..4	25
1.750	21	..4	8	2.1250	42	..4	49
1.750	49	..4	8	2.8750	14	..4	119
1.750	57	..4	8	4.3750	6	..4	34
1.9375	56	..4	54	2.50	7	.34	216
1.9375	60	..4	54	3.250	3	.34	191
1.9375	42	..4	54	2.1250	21	..4	49
1.9375	62	..4	54	5.8750	1	..4	19

26 possibles type (α, β) for 2 level APN-quartic.

conclusion

We have accumulated numerical evidences that seems to prove that all 6-bit APN-functions are known. However, we are still far from being able to produce numerical/theorical proof without further progress.

find a way to finish bent project and ab-APN project with and without space conditions

1. test for APN-extendibility of $(6, 3)$-maps ?
2. maximal degree of an APN ?
3. algorithmic criterion to eliminate bad compoents ?

rush

In dimension 6, if F has type $(1, 4)$ then F is quadratic.